

Spin 1/2 Fermions in the Unitary Limit.II

H. S. Köhler ¹*Physics Department, University of Arizona, Tucson, Arizona 85721, USA***Abstract**

This report concerns the energy of a zero-temperature many-body system of spin $\frac{1}{2}$ fermions in the unitary limit. In a previous report (nucl-th/0705.0944) this energy was determined to be $\xi = 0.24$ in units of the free gas kinetic energy, appreciably lower than most reports giving $\xi \sim 0.45$. In our calculation the 2-body interaction satisfied exactly the unitary limit i.e. infinite scattering length and effective range $r_0 = 0$. In the present report results with $r_0 > 0$ are shown. A strong dependence on the effective range is found. It is for example found that an increase to $r_0 = 1 fm$ increases ξ to ~ 0.4 close to other reports of ξ in the unitary limit. It is concluded that because of the singular character of the unitary limit it is necessary to verify that the interaction actually satisfies unitarity.

The calculations done here in a pp-ladder approximation show a resonance in the *in-medium* interaction close to (and in) the unitary limit. This was already found in the previous work.

1 Introduction

The properties of a dilute fermigas with large scattering length is of considerable theoretical as well as experimental interest.

Several numerical methods have been used to determine ξ with results varying from $\xi \sim 0.5$ to $\xi \sim 0.25$. Extensive references are for example found in [1, 2]

Apart from the interest in the properties of a unitary gas these calculations also provide a test of many-body methods. The Monte Carlo calculations are at least in principle the most accurate and could provide a benchmark.

It is a well-known fact that interactions with large scattering lengths are separable. This paper is a report on results of calculations using separable interactions determined by inverse scattering from phase-shifts of various large scattering lengths a_s and small effective ranges including $a_s = \infty$ and $r_0 = 0$. The energy at zero temperature is calculated from an in-medium effective interaction obtained from a pp-ladder summation of these separable interactions.

Section 2 is a short summary of the method used with numerical results shown in Section 3 and some of the conclusions are summarised in Section 4.

2 Separable Interaction from inverse scattering

The method used here to calculate a separable interaction by inverse scattering has also been used in several previous papers where details can be found.[3, 4, 5, 6] Only a short summary of the expressions is given here. The input in the calculations are phase-shifts, either experimental or otherwise defined as shown below. A rank one separable potential provides a sufficient and in fact precise description of the interaction in the unitary limit. If the phase-shift changes sign such as in the experimental 1S_0 case a rank two potential is necessary. (see ref [3]). In the case of a rank one attractive potential one has

$$V(k, p) = -v(k)v(p) \quad (1)$$

Inverse scattering then yields (e.g. ref [3, 7])

$$v^2(k) = \frac{(4\pi)^2}{k} \sin\delta(k) |D(k^2)| \quad (2)$$

where

$$D(k^2) = \exp \left[\frac{2}{\pi} \mathcal{P} \int_0^\Lambda \frac{k' \delta(k')}{k^2 - k'^2} dk' \right] \quad (3)$$

where \mathcal{P} denotes the principal value and $\delta(k)$ is the phaseshift. Λ provides a cut-off in momentum-space. The effect of the cut-off will be exploited below.

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With $\delta(k) = \pi/2$, the unitary limit, one finds

$$v_u^2(k) = -\frac{(4\pi)^2}{(\Lambda^2 - k^2)^{\frac{1}{2}}} \quad (4)$$

and $v_u^2(k) \rightarrow -\infty$ for $k \rightarrow \Lambda$. (See Fig 2).

For $\Lambda \gg k$ one finds

$$v_u^2(k) \rightarrow -\frac{(4\pi)^2}{\Lambda} \quad (5)$$

In this limit, *but only in this limit*, the unitary interaction will then be a δ -function in coordinate space. And the strength is inversely proportional to the cut-off.

The $V_{low\ k}$ approximation is adequate at low density for some of the cases shown below but NOT in the unitary limit. An effective in-medium Brueckner interaction as defined by a particle-particle ladder summation is used for all the results presented here. Dispersion corrections are expected to be small [6] so that the denominator has only kinetic energies; the effective mass $m^* = 1$. The diagonal elements of the in-medium interaction is then

$$G(k, P) = -\frac{v^2(k)}{1 + I_G(k, P)} \quad (6)$$

with

$$I_G(k, P) = \frac{1}{(2\pi)^3} \int_0^\Lambda v^2(k') \frac{Q(k', P)}{k^2 - k'^2} k'^2 dk' \quad (7)$$

where P is the center of mass momentum and Q the angle-averaged Pauli-operator for pp-ladders (Brueckner approximation). One should note that the angle-averaging is exact here because the denominator is independent of P .

It was shown in ref [4] that this can be rewritten as

$$G(k, P) = -\frac{v^2(k)}{I_{GK}(k, P)} \quad (8)$$

with

$$I_{GK}(k, P) = \frac{1}{(2\pi)^3} \int_0^{2k_f} v^2(k') \frac{Q(k', P) - \mathcal{P}}{k^2 - k'^2} k'^2 dk' + \frac{kv^2(k)}{\tan \delta(k)} \quad (9)$$

Eqs (8, 9) have the advantage over eqs (6, 7) in that the integrand in eq (9) is zero for $k' > 2k_f$ because the factor $Q(k', P) - \mathcal{P}$ is then equal to zero. Consequently, there is no need to resort to a low-momentum (" $V_{low\ k}$ ") approximation here. Although the two sets of equations are numerically identical the latter set simplifies the computing greatly. This is in particular important in the unitary limit in which case the integration can be done analytically to give eq (11).

With $a = \frac{k}{k_f}$ and $y = \frac{P}{2k_f}$ the potential energy per particle PE/A is

$$PE/A = \frac{3k_f^3}{\pi^2} \int_0^1 \left[\int_0^{1-a} 8G(a, y) y^2 dy + \frac{1}{a} \int_{1-a}^{(1-a^2)^{\frac{1}{2}}} 4G(a, y) (1 - y^2 - a^2) y dy \right] a^2 da \quad (10)$$

The kinetic energy per particle, i.e. the uncorrelated fermi-gas energy is given by

$$E_{FG}/A = \frac{3}{10} \frac{\hbar^2}{m} k_f^2.$$

The total energy is expressed in these units by

$$E/A = \xi E_{FG}/A.$$

In the unitary limit ($\delta(k) = \pi/2$) there is a simplification already used in ref [4]. The cutoff Λ can then be chosen large so that eq (5) is valid and with $v_u(k')$ independent of k' one can perform the k' -integration in eq (9) analytically. After dividing by v_u^2 one then finds

$$I_u(a, y) = \frac{k_f}{\pi} \left[1 + y + a * \log \left| \frac{1 + y - a}{1 + y + a} \right| + \frac{1}{2y} (1 - y^2 - a^2) \log \left| \frac{(1 + y)^2 - a^2}{1 - y^2 - a^2} \right| \right]. \quad (11)$$

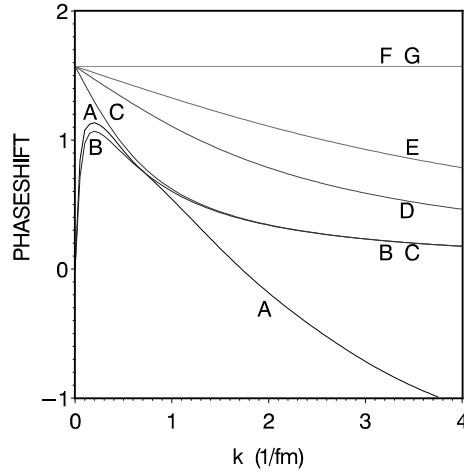


Figure 1: This figure shows the phase-shifts that were used as input for the calculations with the labels defined in the text below.

and

$$G(a, y) = -4\pi[I_u(a, y)]^{-1} \quad (12)$$

This then provides an analytic expression for $G(a, y)$ and as already shown in ref [4] one finds in this case $\xi = 0.24$, explicitly independent of Λ and of density (fermi-momentum) as is to be expected in the unitary limit.

It is of interest that Steele [8], using the effective field theory power counting method arrives at the same expression for $G(a, y)$. But in eq 10 for the potential energy he uses the approximation $G(a, y) \sim G(0, 0)$ and then arrives at $\xi = 4/9$.

3 Numerical Results

The expression (9) was used in all calculations together with the analytic expression (11), valid the in unitary limit.

Calculations were made with phase-shifts defined as follows:

- A: The experimental 1S_0 phases.
- B: Scattering length $a_s = -18.5fm$ and effective range $r_0 = 2.8fm$.
- C: Scattering length $a_s \rightarrow -\infty$ and effective range $r_0 = 2.8fm$.
- D: Scattering length $a_s \rightarrow -\infty$ and effective range $r_0 = 1.0fm$.
- E: Scattering length $a_s \rightarrow -\infty$ and effective range $r_0 = 0.5fm$.
- F: Scattering length $a_s \rightarrow -\infty$ and effective range $r_0 = 0.fm$ and the interaction given by eq. (4).
- G: Scattering length $a_s \rightarrow -\infty$ and effective range $r_0 = 0.fm$ and the interaction given by eq. (5).

Fig 1 shows these phase-shifts as a function of momentum for $k < \Lambda = 4fm$. Notice that phases "A" and "B" are comparable for small momenta, that "B" and "C" overlap except for small momenta and that "F" and "G" are for the unitary limit, $\delta = \frac{\pi}{2}$.

The potentials corresponding to each of these sets of phase-shifts were calculated using the inverse scattering eqs (2) and (3) with the results shown in Fig 2. The potential curves A \rightarrow D with effective ranges $r_0 \leq 1.0$ are (practically) overlapping for $k < 1fm^{-1}$ even though the scattering length varies over a large interval, $-\infty < a_s < -18.5$. Note however the drastic change of the potentials when r_0 changes from 1 to 0.5 to 0

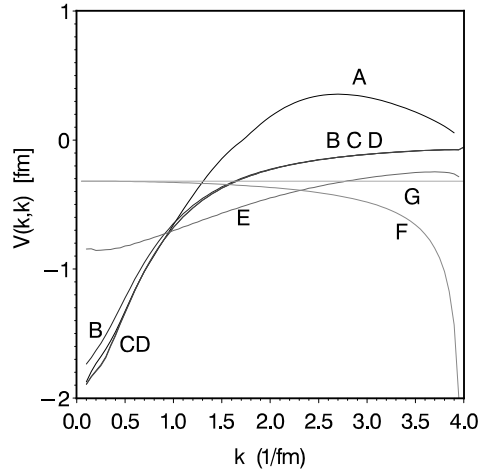


Figure 2: This figure shows the diagonal in momentum-space of the potentials corresponding to the phase-shifts in Fig 1.

(curves labelled D,E and F) while $a_s \rightarrow -\infty$. Note that curve "F" being the exact solution for a separable potential in the unitary limit is uniquely different from all the others.

The total energy ξ in units of the fermigas-energy E_{FG}/A is shown in Fig 3. Our curve "A" for the 1S_0 phases agrees practically exactly with the corresponding curve of ref [2] even though the latter also includes some ring diagrams. Curve "B" with the 1S_0 scattering length and effective range gives a somewhat larger value of ξ . Note that, with a_s held constant at $-\infty$, ξ changes from $\sim 0.4 \rightarrow 0.5$ for $r_0 = 2.8$ to ~ 0.24 for $r_0 = 0$. The value $\xi = 0.24$ independent of k_f is indicated by the straight line. It was already obtained in ref [4] using eq (11). Curve "F" and more so, curve "G", nearly coincide with this analytically obtained result. Curve "G" uses the same large Λ limit of v_u given by eq (5) as in the analytic calculation, while "F" is calculated with $\Lambda = 6 fm^{-1}$. The value of $\xi = 0.24$, given by the straight line, is the unitary limit result from the analytic calculation. It is only in this limit that the analytic integration is possible. The numerical results represented by curves "F" and "G" are necessarily less accurate as discussed below. Those calculations were done only in order to confirm computational consistency.

It was already pointed out in ref [4] that, in the unitary limit, $I_{GK} = 0$ along a line in the (a, y) -plane from $\sim (0.8, 0)$ to $\sim (0.9, 0.3)$. The associated singularity in $G(a, y)$ implies a resonance with the *in-medium* phase-shift $\rightarrow \pi/2$ which appears very unusual. It may however not survive in a higher order calculation, in particular including spectral broadening as in Green's function calculations. The singularity in $G(a, y)$ complicates the numerical integration in eq (10) and an interpolation method was used in ref [4]. The same method was now used for obtaining the "F" and "G" results for ξ . Instead of the analytic expression (11) the numerically integrated eq (9) has now to be interpolated and this results in less accuracy resulting in the oscillations seen in Fig 3 especially for curve "F". Because the analytic solution is more accurate it supersedes the "F" and "G" and an improvement in the numerical calculation to remedy this situation was ruled unnecessary.

4 Conclusions

It is, we believe, conclusively shown that a serious study of a system in the unitary limit has to be done in the limit itself. Any extrapolation is questionable, especially as regards the effective range. The unitary limit is (in a broad sense) singular and unique. This is the necessary conclusion from Fig 3. It is in fact suggested already by studying the phase-shifts shown in Fig 1 and even more so the potential-curves in Fig 2. There is no reason

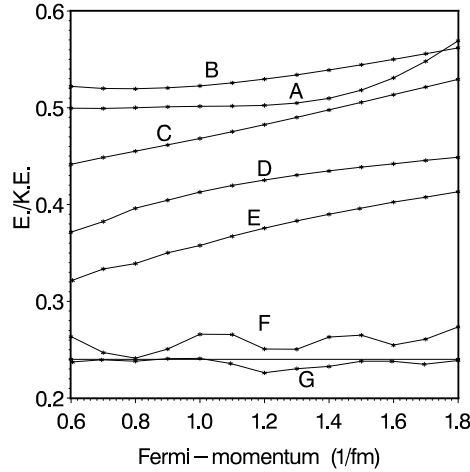


Figure 3: This figure shows the energy (in units of the fermi-gas energy) for various fermi-momenta. The somewhat "wavy" appearance of the curves "F" and "G" (the unitary limit) is because of numerical difficulties described in the text. The straight line is the explicitly density-independent result from an analytic solution of the effective interaction G in the unitary limit.[4]

to believe that this conclusion is unique to the methods used here. The strong dependence of ξ on the effective range r_0 is an important effect and the major result of this investigation.

Using eqs (8) and (9) the cut-off can be chosen arbitrarily large, $\Lambda \gg 2k_f$ and eq (5) is then a valid approximation in the unitary limit. This presents a great simplification resulting in the analytic expression (12) for the effective interaction. This same result was obtained by Steele[8] using power-counting. Eqs (6,7) are numerically equivalent with (8,9) but requires using eq (4) for the interaction resulting in a much more difficult integration with the upper limit equal to Λ where the unitary interaction (4) diverges.

As for the numerical value of ξ it is not claimed to be decided by the zero-temperature ladder approximation used here. Dispersion (mean field) corrections were not included as only kinetic energies were used in the propagators, i.e. $m^* = 1$. This approximation led to eq (9) and eq (11) which provides a great simplification and indeed made the calculations feasible. The dispersion correction is small for the 1S_0 state [6] and although not proven it is expected to be small also in the unitary limit. Dispersion-corrections are related not only to the mean field but also to the in-medium two-body correlations.[6]. It is in general repulsive and would therefore increase the value of ξ . It can not be ruled out that it is larger than expected.

Pairing corrections are important. The critical temperature is relatively high (e.g. [1]) and the boson-fraction is high at zero temperature.

It is generally accepted that Monte Carlo calculations is the best approach to solving the unitary problem. The present investigation is presented only to suggest the importance of using an exactly "unitary interaction".

The calculations were based on using separable interactions obtained by inverse scattering from phase-shifts. Earlier work has shown the near equivalence of this method with the conventional potential approach[3]. It is therefore expected that the conclusions obtained here are not unique to our use of these separable potentials but are in fact general.

In a future publication a report with Green's function techniques allowing a finite temperature calculation including hh-ladders will be reported. These calculations are more extensive than the ladder summations used here. The effects of spectral broadening as well dispersion effects are included.

As already pointed out above there are some similarities of these calculations with those of ref [2] although that work only considers $r_0 \geq 2.54 fm$ for which there is however a semi-quantative agreement between the respective results even though a type of ring-diagrams are included in ref [2] and the interactions are also

different.

After this report was first submitted to the arxiv the author became aware of a work by T. Schaefer et al [9] extending Steele's work. One result of their work was the same value $\xi = 0.24$ referred to above and reported in ref. [4]. They also obtain a strong dependence on effective range as reported here.

I thank Professor Schaefer for kindly bringing this publication to my attention.

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